Working and Saving Informally

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UDP Academic Seminar – August 11, 2022

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Tejada gratefully acknowledges financial support from FONDECYT, grant project No. 11196296.

Presentation Plan

Introduction

Motivation

- Informality is a salient feature in developing economies (La Porta and Shleifer, 2014).
  - In LAC the informal sector represents 41.1% of the GDP and employs between 30 and 80% of the total employment (Gasparini and Tornarolli, 2009).
- Another well-known characteristic of the developing world is the low rate of savings.
  - In LAC, gross domestic savings represent only 17% of the GDP (in high income countries this figure is around 30%).
  - Despite policy efforts to increase the saving levels and good economic conditions, saving rates have remained low in LAC (Reinhardt, 2008).
- The theoretical and empirical literature that independently analyzes the causes and consequences of these two phenomena is vast.
  - The link between informality and savings in developing countries has been less studied and the empirical literature focus on informality $\rightarrow$ savings.
  - Exceptions are Granda and Hamann (2015), Flórez (2017), Esteban-Pretel and Kitao (2022).

This Paper

This paper is a contribution to the recent literature by recognizing the fundamental links between the two phenomena.

- We develop a labor market model where workers can be employed both formally and informally and where agents can save through both formal and informal financial institutions.
- We estimate the model using information of household surveys for Colombia and perform counterfactual simulations to analyze the effect of policy changes.

Questions

1. What is the effect of financial exclusion on savings, informality and inequality?
2. What is the role of informality in inducing/preventing precautionary savings under financial exclusion?

Preview of the Results

- Informal workers face significantly higher costs in adjusting their portfolio toward formal financial assets.
- Workers' transits between formal and informal jobs with some frequency so that the formality state is not a permanent state.
- Spells in informality are characterized by lower saving rates.
- Reaching full financial inclusion of informal workers will increase their saving rate by 10 pp and the overall saving rate by 7 pp.
- To achieve the same improvement in the saving rate with labor market policies would require reducing the proportion of informal wage offers by a huge amount, about 50 pp.
- Full financial inclusion would slightly decrease inequality in consumption and in formal assets.

Literature

- Informality:
  - Albrecht et al. (2009), Bosch and Esteban-Pretel (2012), Charlot et al. (2013) and Bobba et al. (2018) in a DMP type setting.
  - Meghir et al. (2015) in a Burdett-Mortensen type setting.
- Optimal savings with heterogeneous agents:
  - Huggett (1993), Aiyagari (1994) and Krussel and Smith (1998) are classic macro papers. Achdou et.al. (2017) revisited this literature in continous time.
  - Krusell et al. (2010) introduces savings in a DMP setting and Bayer and Walde (2010) does it in continuous time.
  - Rendon (2006) and Lise (2013) introduces savings in a partial equilibrium search models.
- Structural estimation:
  - Flinn and Heckman (1986) and Flinn (2002) estimation of partial equilibrium search models with labor market information.
  - Rendon (2006) and Lise (2013) estimate their model incorporating also data on assets.

The Model

Model Environment

- Time is continuous and the environment is assumed to be stationary.
- Individuals discount the future at ρ and face common probability of death (with Poisson rate θ).
- Individuals are ex-ante homogeneous in every aspect.
- Individuals objective function (Day and Flinn, 2008; Lise, 2013):

$$E_0 \int_0^\infty e^{-(\rho+\theta)t} \frac{c^\delta}{\delta}$$

- The labor market is characterized by three states: non-employment, employment in a formal job, and employment in an informal job.
- Both non-employed and employed are allowed to search for a job (as in Lise, 2013).

Model Environment

- A job offer is a pair wage and type of job: (w, f). Jobs arrive at rate λ^u and $\lambda^e(f)$.
- Wages are draws form $F(w|f)$ and f is a draw from $p(f)$ with $f = \{0, 1\}$.
- Jobs are terminated at exogenous rate $\eta(f)$.
- Two assets: a_1 risk-less *formal asset* with r_1 and a_2 risky *informal asset* with r_2.
- Total wealth $a = a_1 + a_2$ and the share of formal assets $\phi = \frac{a_1}{a}$.
- Convex cost of adjusting the portfolio ϕ: $\frac{\psi^u}{2}\phi^2$ and $\frac{\psi^e(f)}{2}\phi^2$.
- Budget constraint:
$$da = \left[(r_1\phi + r_2(1-\phi))a + i - c - \frac{\psi(f)}{2}\phi^2\right] dt$$
where i is income (Merton, 1975).
- Individuals cannot borrow: $a \geq 0$.

Model Environment

- r_2 follows a Ornstein-Uhlenbeck process:

$$dr_2 = \kappa(\bar{r}_2 - r_2)dt + \sigma dz$$

  z is a standard Brownian motion and therefore r_2 is stationary with $\mathcal{N}\left(\bar{r}_2, \frac{\sigma^2}{2\kappa}\right)$ (Munk and Sorensen, 2010).

- Income process:

$$di = \begin{cases} dq_{\lambda_1^u} \mathbf{I}_1 w_1 + dq_{\lambda_0^u} \mathbf{I}_0 w_0 - b & u \\ dq_{\eta_1} b + dq_{\lambda_1^e} \mathbf{I}_1 w_1' + dq_{\lambda_0^e} \mathbf{I}_0 w_0' - w_1 & f = 1 \\ dq_{\eta_0} b + dq_{\lambda_1^e} \mathbf{I}_1 w_1' + dq_{\lambda_0^e} \mathbf{I}_0 w_0' - w_0 & f = 0 \end{cases}$$

  where $\lambda_f^u = \lambda^u p(f)$, $\lambda_f^e = \lambda^e(f)p(f)$, and $\mathbf{I}_f$ is an indicator variable for acceptable offers.

The steady state value of unemployment is:

$$\tilde{\rho}U(a,r_2) = \max_{0\leq c\leq \bar{c}, 0\leq \phi\leq 1} \Bigg\{ u(c) + \partial_a U(a,r_2)\left[(r_1\phi + r_2(1-\phi))a + b - c - \frac{\psi^u}{2}\phi^2\right]$$
$$+ \partial_{r_2} U(a,r_2)\kappa(\bar{r}_2 - r_2) + \frac{1}{2}\partial^2_{r_2} U(a,r_2)\sigma^2$$
$$+\lambda^u \sum_{f=0}^{1}\left(\int_w \max\{W(a,r_2,w,f) - U(a,r_2), 0\} dF(w|f)p(f)\right)\Bigg\}$$

The steady state value of employment is:

$$\tilde{\rho}W(a,r_2,w,f) = \max_{0\leq c\leq \bar{c}, 0\leq \phi\leq 1} \{u(c) + \epsilon f + \partial_a W(a,f)\left[(r_1\phi + r_2(1-\phi))a\right.$$
$$\left.+b - c - \frac{\psi^e(f)}{2}\phi^2\right] + \partial_{r_2} W(a,r_2,w,f)\kappa(\bar{r}_2 - r_2)$$
$$+ \frac{1}{2}\partial^2_{r_2} W(a,r_2,w,f)\sigma^2 + \delta(f)\left[U(a,r_2) - W(a,r_2,w,f)\right]$$
$$+\lambda^e \sum_{f=0}^{1}\left(\int_{w'} \max\{W(a,r_2,w',f') - W(a,r_2,w,f), 0\} dF(w'|f')p(f')\right)\Bigg\}$$

Solution Method

- Optimal decisions of consumption are characterized by:

$$c^u(a, r_2) = u'^{-1}(\partial_a U(a, r_2))$$
$$c^e(a, r_2, w, f) = u'^{-1}(\partial_a W(a, r_2, w, f))$$

while the optimal portfolio allocation by:

$$\phi^u(a, r_2) = \frac{(r_1 - r_2)a}{\psi^u} \in [0, 1]$$
$$\phi^e(a, r_2, f) = = \frac{(r_1 - r_2)a}{\psi^e(f)} \in [0, 1]$$

- We use a two-step approach to solve for the steady state equilibrium of the model.
  1. Hamilton-Jacobi-Bellman equations: value function iteration and finite difference with an upwind scheme to approximate the derivatives of the value functions (Achdou et al., 2014, 2017).
  2. Kolmogorov Forward equations: simulation approach to compute the invariant distributions of labor market states and of total assets.

Estimation

Data Description

Gran Encuesta Integrada de Hogares (GEIH): Monthly household survey focused on labor market outcomes

- Individual characteristics (gender, age, years of schooling)
- Labor market states:
  - Non-employment (unemployed + non participating).
  - Formal employment (full-time employees who contribute to the social security).
  - Informal employment (full-time informal employees + self-employed working 48+ hours a week).
- Retrospective information on labor market states (yearly):
  - Transitions from non-employment to each type of job.
  - Transitions from employment (aggregated) to non-employment and to each type of job.
- Labor income and weekly hours worked:
  - Real monthly wages (in US dollars of December 2016).

Data Description

Encuesta Longitudinal Colombiana (ELCA): Longitudinal survey that follows $\approx$ 10000 households in rural and urban areas every three years (2010, 2013, and 2016).

- Individual characteristics (gender, age, years of schooling)
- Labor market outcomes except transitions (same definitions as GEIH).
- Savings behavior
  - Average monthly savings (in US dollars of December 2016).
  - Formal savings (formal financial institutions like banks and employees funds/credit unions)
  - Informal savings (cash, group savings, chit funds, etc).

Sample: male, head of households, between 25 and 65 years old, living in urban areas, and without a College degree ("unskilled").

Descriptive Statistics of the Labor Market

Table 1: Descriptive Statistics on Labor Market Outcomes

	Non-Employment	Formal Employment	Informal Employment
Labor Market States			
Proportion	0.151	0.361	0.488
Wages (hundred of US$ of 2016 per month)			
Mean	–	3.420	2.632
Standard Deviation	–	1.524	1.246
Ratio of Average Wages	–	1.299	1.000
Labor Market Yearly Transitions (row=from, col=to)			
Non-Employment	0.075	0.027	0.032
Formal Employment	–	0.287	–
Informal Employment	–	–	0.400
Employment	0.074	0.049	0.056
Sample			
Number Obs. GEIH	9782	23310	31481

Descriptive Statistics of the Saving Behavior

Table 2: Descriptive Statistics on Saving Behavior

	Non-Employment	Formal Employment	Informal Employment
Individuals who's assets are mostly in formal financial institutions			
Proportion	0.214	0.453	0.270
Individuals who save			
Proportion	0.083	0.271	0.186
Savings (hundred of US$ of 2016 per month)			
Mean	0.483	0.561	0.588
Standard Deviation	0.447	0.549	0.791
Sample			
Number Obs. ELCA	170	506	617

Estimation

- We estimate the model primitive parameters using the Method of Simulated Moments (MSM).

$$\hat{\Theta}_{N,T}(W) = \text{argmin}_{\theta} \frac{1}{2} \left[M_N^D - M_T(\Theta) \right]' W_N \left[M_N^D - M_T(\Theta) \right]$$

- Parametric assumption:

$$\log(w)|f \sim \mathcal{N}\left(\mu(f), \sigma(f)\right)$$

- Parameters to estimate:

$$\Theta = \{b, \lambda^u, \lambda^e(1), \lambda^e(0), p(1), \eta(1), \eta(0), \mu(1), \sigma(1), \mu(0), \sigma(0), \psi(1), \psi(0), \kappa, \sigma\}$$

- Fixed parameters:

$$\{\rho, \theta, r_1, \bar{r}_2, \sigma_{r_2}, \delta\}$$

Fixed Parameters

- $\rho = 0.12$. Discount rates for LAC recommended by multilateral development banks 10-12% (Moore et.al., 2020).
- $\theta = 0.013$. Base on Colombia's life expectancy of 77 years (World Bank).
- $r_1 = 0.075$. 10 year Colombian bonds return in 2016.
- $\bar{r}_2 = 0.079$, $\sigma_{r_2} = 0.031$.
    - Eeckhout and Munshi (2010): chit funds in India generate an implicit interest rate that is at most 2.1 times the formal financial system interest rate.
    - Assumption: The interest rate in the Colombian informal financial system would be in the interval $[0, 0.1575]$ the 99% of the time.

$$0.079 \pm 2.576 \times 0.031 \rightarrow r_2 \sim \mathcal{N}\left(0.079, \frac{\sigma^2}{2\kappa} = 0.0009\right)$$

- $\delta = -0.0.53$. Lower bound estimate in Bond, et.al. (2008) for Colombia.

Identification Discussion

- *Labor market dynamics* $(\lambda^u, \lambda^e(1), \lambda^e(0), p(1), \eta(1), \eta(0))$. We use the (pseudo) transition matrix.

$$\lambda^u, p(1) \leftarrow \begin{cases} \Pr[NE \to NE] \\ \Pr[NE \to F] \\ \Pr[NE \to I] \end{cases}$$

$$\lambda^e(1) \leftarrow \begin{cases} \Pr[F \to F|\text{same job}] \\ \Pr[E \to F] \\ \Pr[E \to I] \end{cases} \qquad \eta(1) \leftarrow \begin{cases} \Pr[E \to NE] \\ \Pr[NE] \\ \Pr[F] \\ \Pr[I] \end{cases}$$

$$\lambda^e(0) \leftarrow \begin{cases} \Pr[I \to I|\text{same job}] \\ \Pr[E \to F] \\ \Pr[E \to I] \end{cases} \qquad \eta(0) \leftarrow \begin{cases} \Pr[E \to NE] \\ \Pr[NE] \\ \Pr[F] \\ \Pr[I] \end{cases}$$

Identification Discussion

- *Wages distributions*: $(b, \mu(1), \sigma(1), \mu(0), \sigma(0))$. We use the log-normality assumption and the observed cross-section wages distributions.

$$\mu(1), \sigma(1) \leftarrow \begin{cases} E[w|F] \\ SD[w|F] \end{cases} \qquad \mu(0), \sigma(0) \leftarrow \begin{cases} E[w|I] \\ SD[w|I] \end{cases} \qquad b \leftarrow \begin{cases} P5[w|F] \\ P5[w|I] \end{cases}$$

- *Cost of adjusting the portfolio and the process of r_2*: $(\psi(1), \psi(0), \kappa, \sigma)$. We use moments of the observed distribution of financial assets and the behavior of individual in choosing financial assets to accumulate wealth.

$$\psi(1), \psi(0), \kappa, \leftarrow \begin{cases} \Pr[\phi > 0.5|j] = \Pr[\frac{(r_1 - r_2(\kappa))a}{\psi(j)} > 0.5|j] & i = 0 \; if \; NE, F, I \\ E[da/dt|j] & j = NE, F, I \\ \sigma = 0.031\sqrt{2\kappa} \end{cases}$$

Table 3: Labor Market Parameters

Definition	Parameter	Est. Value	Std. Error
Mobility			
Job offer rate - non-employment	λ^u	0.168	(0.03598)
Job offer rate - formal employment	$\lambda^e(1)$	0.023	(0.00921)
Job offer rate - informal employment	$\lambda^e(0)$	0.030	(0.00673)
Job separation rate - formal employment	$\eta(1)$	0.027	(0.00275)
Job separation rate - informal employment	$\eta(0)$	0.049	(0.00712)
Job Offers Distributions			
Proportion of formal jobs	$p(1)$	0.280	(0.01020)
Mean of wages distribution - formal employment	$\mu(1)$	1.190	(0.01005)
Std.Dev. of wages distribution - formal employment	$\sigma(1)$	0.350	(0.00671)
Mean of wages distribution - informal employment	$\mu(0)$	0.742	(0.01286)
Std. Dev. of wages distribution - informal employment	$\sigma(0)$	0.481	(0.01498)

NOTE: Bootstrap standard errors in parentheses.

Table 4: Financial Parameters

Definition	Parameter	Est. Value	Std. Error
Portfolio Adjustment Cost			
Adjustment cost - non-employment	ψ^u	0.023	(0.00572)
Adjustment cost - formal employment	$\psi^e(1)$	0.024	(0.00504)
Adjustment cost - informal employment	$\psi^e(0)$	0.174	(0.03599)
Informal Assets Returns Process			
Persistence of the rate	κ	0.683	(0.01657)
Standard Deviation of the shock	σ	0.036	(0.02562)
Non-employment Income			
Flow value	b	0.220	(0.05350)

NOTE: Bootstrap standard errors in parentheses.

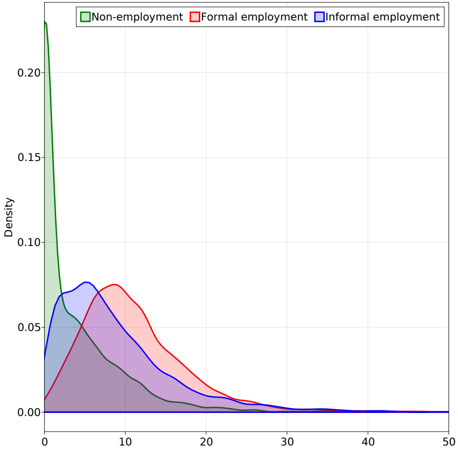
Fit of the Model

Table 5: Moments Fit

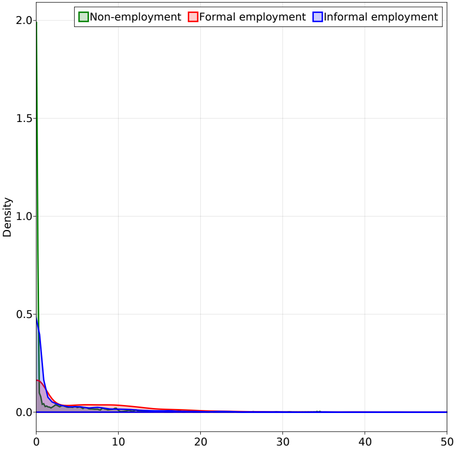
	Data	Model		Data	Model
u	0.151	0.157	$\Pr[e \rightarrow u]$	0.074	0.014
$e(1)$	0.361	0.348	$\Pr[e \rightarrow e(1)]$	0.049	0.002
$e(2)$	0.488	0.495	$\Pr[e \rightarrow e(0)]$	0.056	0.004
$E[w(1)]$	3.420	3.643	$\Pr[\phi > 0.5 \mid u]$	0.214	0.241
$SD[w(1)]$	1.524	1.273	$\Pr[\phi > 0.5 \mid e(1)]$	0.453	0.470
$E[w(0)]$	2.632	2.596	$\Pr[\phi > 0.5 \mid e(0)]$	0.270	0.246
$SD[w(0)]$	1.246	1.287	$E[I_{s>0} \times s \mid u]$	0.040	0.000
$P5[w(1)]$	2.287	2.028	$SD[I_{s>0} \times s \mid u]$	0.183	0.000
$P5[w(0)]$	1.001	1.068	$E[I_{s>0} \times s \mid e(1)]$	0.152	0.220
$\Pr[u \rightarrow u]$	0.075	0.143	$SD[I_{s>0} \times s \mid e(1)]$	0.379	0.360
$\Pr[u \rightarrow e(1)]$	0.027	0.007	$E[I_{s>0} \times s \mid e(0)]$	0.110	0.239
$\Pr[u \rightarrow e(0)]$	0.032	0.020	$SD[I_{s>0} \times s \mid e(0)]$	0.410	0.378
$\Pr[e(1) \rightarrow e(1) \mid same\ job]$	0.287	0.339			
$\Pr[e(0) \rightarrow e(0) \mid same\ job]$	0.400	0.471			

NOTE: $s = da/dt$ is the amount saved and $I_{s>0}$ is an indicator variable that takes the value of 1 if the individual saves a positive amount and zero otherwise.

Steady State Distributions

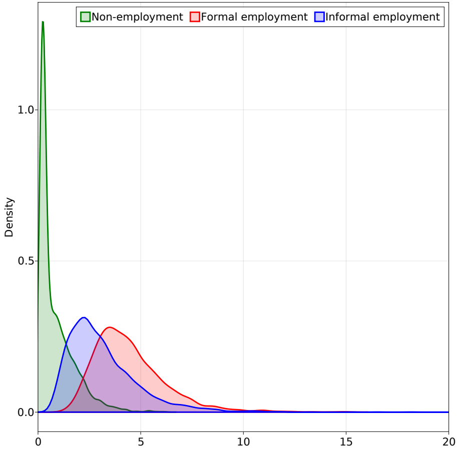


(a) Total Assets

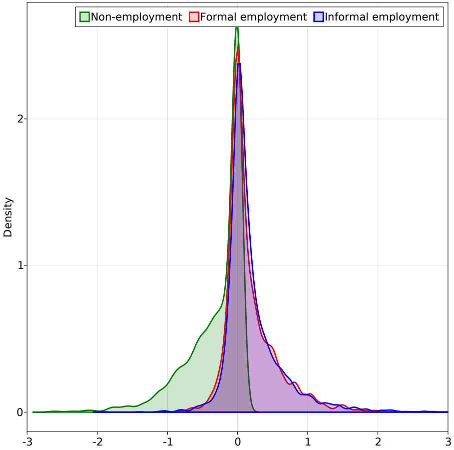


(b) Financial Assets

Steady State Distributions



(c) Consumption



(d) Savings

Counterfactual experiments

Definitions

We perform two sets of counterfactual experiments:

- *Full inclusion of informal workers into the formal financial system*: equal portfolio adjustment costs $\psi^e(0) = \psi^e(1) = 0.024$.
- *Labor market policies that reduce informality*: Proportion of informal job offers drops from the baseline 72% to 20%.

We evaluate the impact on labor market and financial outcomes and on wealth and consumption inequality taking into account the endogenous adjustment in individual's optimal behaviors.

Table 6: Counterfactual Experiments - Labor Market Outcomes

	Benchmark	Financial Inclusion $\psi^e(0) = \psi^e(1) = 0.024$		Lower LM Informality $p(0) = 0.2$	
	Value	Value	Ratio	Value	Ratio
Labor market states					
u	0.157	0.158	1.003	0.134	0.851
$e(1)$	0.348	0.345	0.992	0.765	2.198
$e(0)$	0.495	0.497	1.005	0.102	0.205
Wages					
$E[w\|e(1)]$	3.643	3.618	0.993	3.723	1.022
$E[w\|e(0)]$	2.596	2.628	1.012	2.607	1.004
$E[w\|e(1)]/E[w\|e(0)]$	1.403	1.377	0.981	1.428	1.018

NOTE: Benchmark's values are: $\psi^e(0) = 0.174; \psi^e(1) = 0.024; p(0) = 0.72$. Results are based on simulations of 10.000 individuals.

Table 7: Counterfactual Experiments - Financial Outcomes

	Benchmark	Financial Inclusion $\psi^e(0) = \psi^e(1) = 0.024$		Lower LM Informality $p(0) = 0.2$	
	Value	Value	Ratio	Value	Ratio
		Savings			
$E[s]$	0.113	0.122	1.071	0.121	1.068
$E[s \mid e(1)]$	0.182	0.182	1.000	0.188	1.037
$E[s \mid e(0)]$	0.205	0.226	1.105	0.207	1.011
		Assets in Formal Institutions			
$E[\phi a]$	3.462	4.104	1.186	4.174	1.206
$E[\phi a \mid e(1)]$	5.208	5.238	1.006	4.883	0.938
$E[\phi a \mid e(0)]$	2.852	4.166	1.461	2.514	0.881
		Total Assets			
$E[a]$	8.650	8.681	1.004	8.945	1.034
$E[a \mid e(1)]$	11.011	10.789	0.980	10.204	0.927
$E[a \mid e(0)]$	8.715	8.880	1.019	7.709	0.885

NOTE: Benchmark's values are: $\psi^e(0) = 0.174$; $\psi^e(1) = 0.024$; $p(0) = 0.72$. Results are based on simulations of 10.000 individuals.

Table 8: Counterfactual Experiments - Inequality

	Benchmark	Financial Inclusion $\psi^e(0) = \psi^e(1) = 0.024$		Lower LM Informality $p(0) = 0.2$	
	Value	Value	Ratio	Value	Ratio
Total Assets					
$GE(0)$	2.990	3.083	1.031	2.967	0.992
$GE(1)$	0.330	0.327	0.991	0.275	0.833
$GE(2)$	0.349	0.341	0.977	0.272	0.778
Assets in Formal Institutions					
$GE(0)$	6.340	6.080	0.959	6.158	0.971
$GE(1)$	0.581	0.450	0.775	0.413	0.710
$GE(2)$	1.434	1.159	0.808	1.047	0.730
Consumption					
$GE(0)$	0.285	0.284	0.995	0.261	0.917
$GE(1)$	0.204	0.201	0.984	0.169	0.830
$GE(2)$	0.200	0.193	0.966	0.153	0.766

NOTE: Benchmark's values are: $\psi^e(0) = 0.174; \psi^e(1) = 0.024; p(0) = 0.72$. Results are based on simulations of 10.000 individuals.

Concluding remarks and next steps

Concluding Remarks

- Workers in many low- and middle-income countries are characterized by high probability to work informally and they have low savings, frequently allocated outside formal financial institutions.
- We develop an environment able to integrate the behaviors leading to both phenomena.
- Our environment has two types of jobs (formal and informal) and a portfolio choice between two types of assets, a formal risk-less asset and an informal risky assets.
- We use data from Colombia to estimate the model that are complete enough to characterize both labor market and saving behaviors.
- Estimation results show that informal workers face higher costs of saving in formal financial assets and that formality state is not a permanent state of a typical individual labor market career.

Concluding Remarks

- We perform two counterfactual experiments using the estimated model so as to evaluate policy changes in an equilibrium setting.
- Financial inclusion of informal workers result in a significant increase in the informal and the overall saving rate.
- A massive reduction of the proportion of informal job offers is able to just barely generate a saving rate similar to the one obtained with full financial inclusion.
- Full financial inclusion slightly decreases inequality in consumption and in formal assets but less so than the labor market policy.

Next Steps

We are working in improving some limitations of the current model environment.

- Utility value of working formally in a similar fashion of Dey and Flinn (2008) and Conti et.al. (2018):

$$E_0 \int_0^{\infty} e^{-\tilde{\rho} t} \left[u(c) + \epsilon f \right]$$

  where ϵ is a non-negative scalar and f is an indicator variable that takes the value of 1 if the individual is working formally and 0 otherwise.

- Additional policy variable: pay-roll tax paid only by individual who are working formally.

$$da = \begin{cases} \left[(r_1\phi + r_2(1-\phi))a + b - c - \frac{\psi^u}{2}\phi^2 \right] dt & \text{if non-employed} \\ \left[(r_1\phi + r_2(1-\phi))a + w(f)(1-\tau f) - c - \frac{\psi^{e(f)}}{2}\phi^2 \right] dt & \text{if employed}(f = 0, 1) \end{cases}$$

- Possibility of borrowing from formal and informal financial institutions while maintaining the incomplete markets assumption.

$$a \geq \underline{a} = -b / r_2^{\max}$$

Thank You!!

Additional slides

- Value functions iteration with a discretized state space and an upwind finite difference method to approximate the derivatives (Achdou et.al., 2017).
- Define $W_{i,j,k,f}$ and $U_{i,j}$ for the grids a_i, $r_{2,j}$, w_k.

$$\partial_a U(a, r_2) \approx \begin{cases} \frac{U_{i+1,j} - U_{i,j}}{a_{i+1} - a_i} & da > 0 \\ \frac{U_{i,j} - U_{i-1,j}}{a_i - a_{i-1}} & da < 0 \end{cases}$$

$$\partial_a W(a, r_2, w, f) \approx \begin{cases} \frac{W_{i+1,j,k,f} - W_{i,j,k,f}}{a_{i+1} - a_i} & da > 0 \\ \frac{W_{i,j,k,f} - W_{i-1,j,k,f}}{a_i - a_{i-1}} & da < 0 \end{cases}$$

- Upwind to approximation $\partial_a U(a, r_2)$ and $\partial_a W(a, r_2, w, f)$
- The upwind approximation $\partial^2_{r_2} U(a, r_2)$ and $\partial_{r_2} W(a, r_2, w, f)$ is similar, use *forward difference* when $dr_2 > 0$ and *backward difference* when $dr_2 < 0$.

Solution Method

- We use again finite differences to approximate the second derivative.

$$\partial^2_{r_2} U(a, r_2) \approx \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{(r_{2,j+1} - r_{2,j})^2}$$

$$\partial^2_{r_2} W(a, r_2, w, f) \approx \frac{W_{i,j+1,k,f} - 2W_{i,j,k,f} + W_{i,j-1,k,f}}{(r_{2,j+1} - r_{2,j})^2}$$

- Boundary conditions in a-dimension are needed for the backward approximation:

$$\begin{aligned}\partial_a U(\underline{a}, r_2) &= u'(r_1\phi(\underline{a}, r_2, 0) + r_2(1 - \phi(\underline{a}, r_2, 0)))\underline{a} + b - c^u(\underline{a}, r_2) \\ &\quad - \frac{\psi^u}{2}\phi(\underline{a}, r_2, 0)^2) \\ \partial_a W(\underline{a}, r_2, w, f) &= u'(r_1\phi(\underline{a}, r_2, w, f) + r_2(1 - \phi(\underline{a}, r_2, w, f)))\underline{a} + w - c^u(\underline{a}, r_2, w, f) \\ &\quad - \frac{\psi^e(f)}{2}\phi(\underline{a}, r_2, w, f)^2)\end{aligned}$$

Solution Method

- Boundary conditions in r_2-dimension:

$$\partial_{r_2} U(a, \underline{r_2}) = 0 \Rightarrow U_{i,0} = U_{i,1}$$

$$\partial_{r_2} U(a, \bar{r}_2) = 0 \Rightarrow U_{i,J+1} = U_{i,J}$$

$$\partial_{r_2} W(a, \underline{r_2}, w, f) = 0 \Rightarrow W_{i,0,k,f} = W_{i,1,k,f}$$

$$\partial^2_{r_2} W(a, \bar{r}_2, w, f) = 0 \Rightarrow W_{i,J+1,k,f} = W_{i,J,k,f}$$